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Dedicated to our friend Aleš Pultr on the occasion of his 80th birthday.

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ABSTRACT

In the standard approach to pointfree topology via frames/locales, a topological space is represented by its frame of open subsets. The aim of this note is to propose an alternative approach, by representing a topological space by a frame of continuous functions with values in what we call a *topological frame*. A prime example of such a familiar representation is the frame of lower semicontinuous $[0, \infty]$ -valued functions on a topological space.

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1. Introduction and preliminaries

Let us recall that a frame is a complete lattice \mathbb{F} (the bottom, resp. top, of which will be generally denoted as 0, resp. 1 (except in the specific examples treated in section 3.2)), in which for all $x \in \mathbb{F}$ and all $A \subseteq \mathbb{F}$ the distributive law $x \wedge (\bigvee A) = \bigvee \{x \wedge a \mid a \in A\}$ holds. (If we want to specify the order \leq on

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\mathbb{F} , we will sometimes denote the frame by (\mathbb{F}, \leq) . A map $h : \mathbb{F}_1 \rightarrow \mathbb{F}_2$ between frames \mathbb{F}_1 and \mathbb{F}_2 is called a frame homomorphism if it preserves all joins and finite meets (including bottom and top). The resulting category is denoted \mathbf{Frm} and its opposite category \mathbf{Frm}^{op} is called the category of locales. Frames or locales form the basis of so-called pointfree topology. For categorical terminology we refer to [1] and we will use the monograph [8] as our standard reference regarding pointfree topology (see also [4] or [10]). In the sequel, the category of sets (and maps), resp. of topological spaces (and continuous maps), will be denoted by \mathbf{Set} , resp. \mathbf{Top} .

The usual starting point of pointfree topology is representing a given topological space X within the category \mathbf{Frm} by its frame $\mathcal{O}(X)$ of open subsets (which indeed is a frame for the order provided by subset inclusion). Moreover, if $f : X \rightarrow Y$ is a continuous map between topological spaces, it is obvious that $\mathcal{O}(f) : \mathcal{O}(Y) \rightarrow \mathcal{O}(X), U \mapsto f^{-1}(U)$ is a frame homomorphism and that we in this way obtain a functor $\mathcal{O} : \mathbf{Top} \rightarrow \mathbf{Frm}^{\text{op}}$. One of the basic goals of pointfree topology is trying to answer the question of “how much information about the topological space can be recovered from its frame of opens $\mathcal{O}(X)$, without making use of the points of X ”. This pointfree topology, i.e. study of properties of frames/locales and relating them back to the classical (= pointed) case of topological spaces, has proved to be a rich and surprising topic over the last decades. The passage from spaces to frames/locales not only drastically enhances the scope, but the ability to use methods coming from (universal) algebra on the frame side also provides a more geometric insight into topology, with truly remarkable surprises along the way like e.g. a choice free proof of the Tychonoff theorem for frames/locales (see e.g. [4,8,10]). We also note in passing that going back from the pointfree to the classical setting is done via the so-called spectrum functor $\Sigma : \mathbf{Frm}^{\text{op}} \rightarrow \mathbf{Top}$, being a right adjoint to \mathcal{O} . Again we refer to [8] for more details (the definition of Σ will be recalled further on in this note, where it is needed).

On the other hand approach theory (see [7]), with totally different motivations, hinges on the representation of a topological space X by the set \mathcal{L}_X of lower-semicontinuous $[0, \infty]$ -valued functions on X , which form a frame with respect to the pointwise order on \mathcal{L}_X (as derived from the pointwise order on $[0, \infty]^X$). If we denote the space $[0, \infty]$ endowed with its Scott topology by \mathbb{P} , resp. the two-point space $\{0, 1\}$ with its Scott topology by \mathbb{S} (i.e. \mathbb{S} is in fact the Sierpinski space with $\{1\}$ the only non-trivial open subset), it is easy to see that $\mathcal{L}_X = \mathbf{Top}(X, \mathbb{P})$ but also, using indicator functions, that $\mathcal{O}(X)$ can be identified with $\mathbf{Top}(X, \mathbb{S})$ (moreover, when one equips $\mathbf{Top}(X, \mathbb{S})$, with the pointwise order, we even have that $\mathcal{O}(X)$ and $\mathbf{Top}(X, \mathbb{S})$ are isomorphic as frames).

This begs the following natural question for a given frame \mathbb{F} , endowed with a suitable topology: “how much/which topological information about a topological space X can be captured by the function frame $\mathbf{Top}(X, \mathbb{F})$ (endowed with pointwise order), and how does this depend on the nature of \mathbb{F} ”? Let us first observe that for recovering the space X from the frame $\mathbf{Top}(X, \mathbb{F})$, the usual spectrum functor will not do the trick: it is an easy exercise to check that for a sober topological space X we have that $\Sigma \mathbf{Top}(X, \mathbb{P})$ is homeomorphic to $X \times]0, \infty]$ and not X !

The aim of this note is to investigate how and to what extent we can get rid of the factor $]0, \infty]$ (in the case that $\mathbb{F} = \mathbb{P}$). In section 2, we set up the framework of topological frames \mathbb{F} , and the relevant adjunction between \mathbf{Top} and the (opposite of the) comma category \mathbb{F}/\mathbf{Frm} for a given \mathbb{F} . In section 3, we then mainly turn to the study of the associated notion of soberness. Our main result Theorem 3.5 provides conditions on \mathbb{F} ensuring that a Hausdorff topological space is \mathbb{F} -sober. These conditions are fulfilled as soon as \mathbb{F} is a chain with $0 \neq 1$ endowed with the Scott topology (Corollary 3.7). Further, in section 3.2, we compute \mathbb{F} -spectra of \mathbb{F} -function frames for various spaces X and frames \mathbb{F} , exhibiting a number of spaces that are not \mathbb{F} -sober, and showing in particular that the Hausdorff condition above cannot be relaxed to classical soberness. Finally, in section 3.3 we discuss the relation between the notion of \mathbb{F} -soberness and that of \mathbb{F} -fuzzy soberness as considered in [16].

2. Topological frames and function frames

2.1. The category of topological frames

We begin by introducing the concept of a topological frame, which will play a central role in the sequel as we are going to represent a topological space X by a frame of functions on X with values in such a topological frame. Together with the obvious choice of morphisms they form a category, of which we list some useful properties in this section.

Definition 2.1. Let (\mathbb{F}, \leq) be a frame endowed with a topology $\mathcal{T}_{\mathbb{F}}$. We call $(\mathbb{F}, \leq, \mathcal{T}_{\mathbb{F}})$ a topological frame provided that the operations

$$\wedge : \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F} : (a, b) \longmapsto a \wedge b$$

and

$$\sup_{i \in I} : \mathbb{F}^I \longrightarrow \mathbb{F} : (a_i)_{i \in I} \longmapsto \sup_{i \in I} a_i$$

are continuous. (We will also simply write $(\mathbb{F}, \mathcal{T}_{\mathbb{F}})$ or \mathbb{F} to denote a topological frame.)

We call a subset $U \subseteq \mathbb{F}$ *sup-inaccessible* if $\sup_{i \in I} a_i \in U$ implies the existence of some $j \in I$ with $a_j \in U$. We call U *directed sup-inaccessible* if the same condition holds, but only for directed suprema.

Proposition 2.2. Let \mathbb{F} be a frame endowed with a topology $\mathcal{T}_{\mathbb{F}}$ such that every open set in $\mathcal{T}_{\mathbb{F}}$ is a sup-inaccessible filter. Then $(\mathbb{F}, \mathcal{T}_{\mathbb{F}})$ is a topological frame.

Proof. For $U \in \mathcal{T}_{\mathbb{F}}$, we have

$$\begin{aligned} \sup_I^{-1}(U) &= \{(a_i)_i \in \mathbb{F}^I \mid \sup_{i \in I} a_i \in U\} \\ &= \{a = (a_i)_i \in \mathbb{F}^I \mid \exists j \in I : \text{pr}_j(a) \in U\} \\ &= \bigcup_{i \in I} \{a \in \mathbb{F}^I \mid \text{pr}_i(a) \in U\} = \bigcup_{i \in I} p_i^{-1}(U). \end{aligned}$$

Since U is a filter, it also follows that

$$\begin{aligned} \wedge^{-1}(U) &= \{(a, b) \in \mathbb{F}^2 \mid a \wedge b \in U\} \\ &= \{(a, b) \in \mathbb{F}^2 \mid a, b \in U\} = U \times U. \quad \square \end{aligned}$$

Every frame \mathbb{F} can be endowed with the Scott topology, for which a subset $U \subseteq \mathbb{F}$ is Scott open if it is an upset which is directed sup-inaccessible.

Example 2.3.

- (1) Any complete chain \mathbb{F} endowed with the Scott topology is a topological frame.
- (2) Let \mathbb{S} be the frame $\mathbb{S} = [2] = \{0, 1\}$ with the Scott topology, which reduces to the Sierpinski topology for which $\{1\}$ is open and $\{0\}$ is non-open. By (1) this is a topological frame.
- (3) The frame $\mathbb{S} \times \mathbb{S}$ endowed with the Scott topology is topological.

- (4) The frame $\mathbb{F} = [2] \times [2]$ endowed with the topology for which $\emptyset, \{(1, 1)\}$ and \mathbb{F} are the only open sets is not topological.

If the order and topology of a topological frame $(\mathbb{F}, \leq, \mathcal{T}_{\mathbb{F}})$ are clear from the context, we will simply write \mathbb{F} .

Definition 2.4. Let $(\mathbb{F}_1, \leq_1, \mathcal{T}_1)$ and $(\mathbb{F}_2, \leq_2, \mathcal{T}_2)$ be topological frames. A map $f : \mathbb{F}_1 \rightarrow \mathbb{F}_2$ is called a *topological frame morphism* if $f : (\mathbb{F}_1, \mathcal{T}_1) \rightarrow (\mathbb{F}_2, \mathcal{T}_2)$ is continuous and $f : (\mathbb{F}_1, \leq_1) \rightarrow (\mathbb{F}_2, \leq_2)$ is a frame homomorphism.

We denote **TopFrm** the category with topological frames as objects and topological frame morphisms as morphisms.

There are natural forgetful functors between the categories **TopFrm**, **Frm**, **Top** and **Set**, forgetting either the topology, the order or both, given in the following diagram.

$$\begin{array}{ccc}
 \mathbf{TopFrm} & \xrightarrow{\mathcal{U}_T} & \mathbf{Top} \\
 \mathcal{U}_F \downarrow & \searrow \mathcal{U} & \downarrow \mathcal{U}_{\mathbf{Top}} \\
 \mathbf{Frm} & \xrightarrow{\mathcal{U}_{\mathbf{Frm}}} & \mathbf{Set}
 \end{array}$$

It is known that the functor $\mathcal{U}_{\mathbf{Top}}$ is topological and $\mathcal{U}_{\mathbf{Frm}}$ has a left adjoint (see e.g. [4,8] for a detailed construction of this left adjoint, i.e. the construction of “free frames”). Moreover it is proved in [4] that the obtained adjunction formed by this free functor and $\mathcal{U}_{\mathbf{Frm}}$ is monadic (see also [12]).

The proof of the following useful fact essentially goes along the same lines as a result in [9].

Proposition 2.5. \mathcal{U}_F is topological.

Proof. Let $(f_i : F \rightarrow (L_i, \mathcal{T}_i))_{i \in I}$ be a source of frame homomorphism with the L_i , $i \in I$ being topological frames. Take J to be any index set and endow F with the initial topology for the original source when disregarding the lattice structure. When using \sup as a supremum operator in F and $\sup_J^{(i)}$ as a supremum operator on L_i for $i \in I$, the following diagram commutes for any $i \in I$.

$$\begin{array}{ccc}
 (F, \mathcal{T}_{in})^J & \xrightarrow{\bigotimes_J f_i} & (L_i, \mathcal{T}_i)^J \\
 \sup_J \downarrow & & \downarrow \sup_J^{(i)} \\
 (F, \mathcal{T}_{in}) & \xrightarrow{f_i} & (L_i, \mathcal{T}_i)
 \end{array}$$

Indeed, for $(a_j)_j \in F^J$, we have that

$$(\sup_J^{(i)} \circ \bigotimes_J f_i)((a_j)_j) = \sup_{j \in J}^{(i)} (f_i(a_j)_j) = f_i(\sup_{j \in J} a_j) = (f_i \circ \sup_J)((a_j)_j)$$

since f_i is a frame homomorphism. Since the source $(f_i)_i$ is initial in **Top** and $\sup_J^{(i)} \circ \bigotimes_J f_i = f_i \circ \sup_J$ is continuous for all $i \in I$, we have that \sup_J is also continuous.

It follows analogously that the meet operator is also continuous. Hence the source has an initial lift. This lift is unique, since it is unique in \mathbf{Top} . \square

Using some well-known standard category-theoretic results from [1,3], we immediately can draw the following conclusions:

Corollary 2.6. *\mathcal{U} is topologically algebraic and hence is faithful and is a right adjoint (i.e. has a left adjoint).*

Corollary 2.7. *An arbitrary product of topological frames is topological.*

For the sake of completeness, we recall Wyler's Taut Lift Theorem ([14], also see [1]) and give a construction of a left adjoint G of \mathcal{U}_T such that $\mathcal{U}_F G = F\mathcal{U}_{\mathbf{Top}}$.

Theorem 2.8 (Taut Lift Theorem ([14], also see [1])). *We consider a commutative diagram of functors*

$$\begin{array}{ccc} A & \xrightarrow{G} & B \\ U \downarrow & & \downarrow V \\ X & \xrightarrow{J} & Y \end{array}$$

such that J has a left adjoint $H \dashv_{\delta}^{\gamma} J$ and U is topological. Then G has a left adjoint $F \dashv_{\varepsilon}^{\eta} G$ with $UF = HV$ if and only if G maps U -initial sources to V -initial sources.

When applying this theorem to our situation, a reflection of (X, \mathcal{T}) in \mathbf{Top} along \mathcal{U}_T is given as follows. Consider the \mathcal{U}_T -structured source

$$\mathfrak{S}_X = (X \xrightarrow{f_i} \mathcal{U}_T \mathbb{F}_i)$$

consisting of all pairs (f_i, \mathbb{F}_i) with $\mathbb{F}_i \in |\mathbf{TopFrm}|$ and $f_i \in \mathbf{Top}(X, \mathcal{U}_T \mathbb{F}_i)$.

Then

$$\left(F\mathcal{U}_{\mathbf{Top}} X \xrightarrow{F\mathcal{U}_{\mathbf{Top}} f_i} F\mathcal{U}_{\mathbf{Top}} \mathcal{U}_T \mathbb{F}_i = F\mathcal{U}_{\mathbf{Frm}} \mathcal{U}_F \mathbb{F}_i \xrightarrow{\gamma_{\mathcal{U}_F \mathbb{F}_i}} \mathcal{U}_F \mathbb{F}_i \right)_i$$

is a \mathcal{U}_F -structured source in \mathbf{Frm} . Hence it has a unique initial lift

$$\mathfrak{T}_X = \left(\mathbb{F}_X \xrightarrow{\tilde{f}_i} \mathbb{F}_i \right)_i$$

with $\mathcal{U}_F \tilde{f}_i = \gamma_{\mathcal{U}_F \mathbb{F}_i} \circ F\mathcal{U}_{\mathbf{Top}} f_i$.

Since the source $\mathcal{U}_T \mathfrak{T}_X$ is also initial in \mathbf{Top} and the following diagram commutes for any i

$$\begin{array}{ccccccc} \mathcal{U}_{\mathbf{Top}} X & \xrightarrow{\mu_{\mathcal{U}_{\mathbf{Top}} X}} & \mathcal{U}_{\mathbf{Frm}} F\mathcal{U}_{\mathbf{Top}} X & \equiv & \mathcal{U}_{\mathbf{Frm}} \mathcal{U}_F \mathbb{F}_i & \equiv & \mathcal{U}_{\mathbf{Top}} \mathcal{U}_T \mathbb{F}_X \\ \mathcal{U}_{\mathbf{Top}} f_i \downarrow & & \downarrow \mathcal{U}_{\mathbf{Frm}} \mathcal{U}_F \tilde{f}_i & & \downarrow \mathcal{U}_{\mathbf{Frm}} \mathcal{U}_F \tilde{f}_i & & \downarrow \mathcal{U}_{\mathbf{Top}} \mathcal{U}_T \tilde{f}_i \\ \mathcal{U}_{\mathbf{Top}} \mathcal{U}_T \mathbb{F}_i & \equiv & \mathcal{U}_{\mathbf{Frm}} \mathcal{U}_F \mathbb{F}_i & \equiv & \mathcal{U}_{\mathbf{Frm}} \mathcal{U}_F \mathbb{F}_i & \equiv & \mathcal{U}_{\mathbf{Top}} \mathcal{U}_T \mathbb{F}_i \end{array}$$

Hence a morphism $\eta_X \in \mathbf{Top}(X, \mathcal{U}_T \mathbb{F}_X)$ exists with $\mathcal{U}_{\mathbf{Top}}(\eta_X) = \mu_{\mathcal{U}_{\mathbf{Top}} X}$. Then (\mathbb{F}_X, η_X) is the reflection of X along \mathcal{U}_T .

2.2. Function frames

Given a topological frame \mathbb{F} , we now come to introducing the frame $\mathbf{Top}(X, \mathbb{F})$ that we will use to represent a topological space X in our approach. We use the notation $\mathbf{Fun}(X, \mathbb{F})$ for the frame of maps from X to \mathbb{F} endowed with the pointwise order.

Proposition 2.9. *Let \mathbb{F} be a topological frame and let X be a topological space. The subset of continuous functions $\mathbf{Top}(X, \mathbb{F}) \subseteq \mathbf{Fun}(X, \mathbb{F})$ is a subframe.*

Proof. First note that the constant top and bottom functions are continuous, so the top and bottom of $\mathbf{Fun}(X, \mathbb{F})$ are contained in $\mathbf{Top}(X, \mathbb{F})$. Next consider $f, g \in \mathbf{Top}(X, \mathbb{F})$. The pointwise infimum $f \wedge g$ is obtained as the composition

$$X \xrightarrow{(f,g)} \mathbb{F} \times \mathbb{F} \xrightarrow{\wedge} \mathbb{F}$$

of continuous maps. Consider a family $(f_i)_{i \in I}$ with $f_i \in \mathbf{Top}(X, \mathbb{F})$. The pointwise supremum $\sup_{i \in I} f_i$ is obtained as the composition

$$X \xrightarrow{(f_i)_i} \mathbb{F}^I \xrightarrow{\sup_i} \mathbb{F}$$

of continuous maps. This finishes the proof. \square

Note that there is a natural frame homomorphism $\Gamma_X : \mathbb{F} \rightarrow \mathbf{Top}(X, \mathbb{F}) : a \mapsto c_a$ where $c_a : X \rightarrow \mathbb{F} : x \mapsto a$ is the constant function with value a . Let \mathbb{F}/\mathbf{Frm} denote the comma category. Objects of \mathbb{F}/\mathbf{Frm} are frame homomorphisms $\mathbb{F} \rightarrow L$ and are called \mathbb{F} -frames. Hence, we can consider Γ_X as an \mathbb{F} -frame. A morphism between $(L, \gamma_L : \mathbb{F} \rightarrow L)$ and $(L', \gamma_{L'} : \mathbb{F} \rightarrow L')$ in \mathbb{F}/\mathbf{Frm} is a frame homomorphism $h : L \rightarrow L'$ such that $h\gamma_L = \gamma_{L'}$.

Definition 2.10. Let \mathbb{F} be a topological frame. For a topological space X , the \mathbb{F} -frame $\mathcal{O}_{\mathbb{F}}(X) = \Gamma_X$ is called the \mathbb{F} -function frame of X .

A continuous map $\varphi : X \rightarrow X'$ between topological spaces naturally gives rise to a frame homomorphism

$$\mathcal{O}_{\mathbb{F}}(\varphi) : \mathbf{Top}(X', \mathbb{F}) \longrightarrow \mathbf{Top}(X, \mathbb{F}) : f \longmapsto f\varphi$$

which satisfies $\mathcal{O}_{\mathbb{F}}(\varphi)\Gamma_{X'} = \Gamma_X$. We thus obtain a functor

$$\mathcal{O}_{\mathbb{F}} : \mathbf{Top} \longrightarrow \mathbb{F}/\mathbf{Frm}^{\text{op}}.$$

2.3. The \mathbb{F} -spectrum

Let \mathbb{F} be a topological frame with topology $\mathcal{T}_{\mathbb{F}}$ and let $L = (L, \gamma_L : \mathbb{F} \rightarrow L)$ be an \mathbb{F} -frame. The \mathbb{F} -frame $\mathbb{F} = (\mathbb{F}, 1_{\mathbb{F}} : \mathbb{F} \rightarrow \mathbb{F})$ is the initial object in \mathbb{F}/\mathbf{Frm} .

Consider the set $\mathbf{Frm}_{\mathbb{F}}(L, \mathbb{F}) = (\mathbb{F}/\mathbf{Frm})(L, \mathbb{F})$ of frame homomorphisms $\psi : L \rightarrow \mathbb{F}$ with $\psi\gamma_L = 1_{\mathbb{F}}$ and the source of maps

$$(\text{ev}_l : \mathbf{Frm}_{\mathbb{F}}(L, \mathbb{F}) \longrightarrow \mathbb{F} : f \longmapsto f(l))_{l \in L}.$$

Definition 2.11. The \mathbb{F} -spectrum of L is the set $\text{Spec}_{\mathbb{F}}(L) = \text{Frm}_{\mathbb{F}}(L, \mathbb{F})$ endowed with the initial topology for the source $(\text{ev}_l)_{l \in L}$.

Proposition 2.12. A frame homomorphism $\varphi : L \rightarrow L'$ gives rise to a continuous map

$$\text{Spec}_{\mathbb{F}}(\varphi) : \text{Frm}_{\mathbb{F}}(L', \mathbb{F}) \longrightarrow \text{Frm}_{\mathbb{F}}(L, \mathbb{F}) : f \longmapsto f\varphi.$$

Proof. This follows by definition of the initial topology on $\text{Frm}_{\mathbb{F}}(L, \mathbb{F})$, since the compositions $\text{ev}_l \text{Spec}_{\mathbb{F}}(\varphi) = \text{ev}_{\varphi(l)} : \text{Frm}_{\mathbb{F}}(L', \mathbb{F}) \longrightarrow \mathbb{F}$ are continuous for the initial topology on $\text{Frm}_{\mathbb{F}}(L', \mathbb{F})$. \square

We thus obtain a functor

$$\text{Spec}_{\mathbb{F}} : \mathbb{F}/\text{Frm}^{\text{op}} \longrightarrow \text{Top}.$$

Note that for $L = \text{Top}(X, \mathbb{F})$,

$$\begin{aligned} \text{Frm}_{\mathbb{F}}(L, \mathbb{F}) &= \{\psi : L \rightarrow \mathbb{F} \text{ frame homomorphism} \mid \psi \Gamma_X = 1_{\mathbb{F}}\} \\ &= \{\psi : L \rightarrow \mathbb{F} \text{ frame homomorphism} \mid \forall a \in \mathbb{F} : \psi(c_a) = a\}. \end{aligned}$$

For $\mathbb{F} = \mathbb{S}$, we have

$$\begin{aligned} \text{Frm}_{\mathbb{S}}(L, L') &= \{\psi : L \rightarrow L' \text{ frame homomorphism} \mid \psi \gamma_L(0) = \perp_{L'}, \psi \gamma_L(1) = \top_{L'}\} \\ &= \text{Frm}(L, L'). \end{aligned}$$

For a frame L , we recall two (homeomorphic) definitions of the spectrum of L , namely

- $\text{Spec}_{\wedge}(L) := (\{a \in L \mid a \text{ is meet-irreducible}\}, \{\Sigma'_a \mid a \in L\})$, where $\Sigma'_a = \{p \mid a \not\leq p\}$,
- $\Sigma L := (\text{Frm}(L, \mathbb{S}), \{\Sigma_a \mid a \in L\})$, where $\Sigma_a = \{h \in \text{Frm}(L, \mathbb{S}) \mid h(a) = 1\}$.

Since for any frame L there exists a unique frame morphism $\mathbb{S} \rightarrow L$, we can see that the definitions of frame and \mathbb{S} -frame coincide. Moreover the spectrum ΣL of L is the same (as a set) as the \mathbb{S} -spectrum of L . The following proposition shows that both spectra are the same when considered as topological spaces.

Proposition 2.13. The topological spaces $\text{Spec}_{\wedge}(L)$, $\text{Spec}_{\mathbb{S}}(L)$ and ΣL are homeomorphic.

Proof. Consider the following maps

$$\begin{aligned} \psi_1 : \Sigma L &\longrightarrow \text{Spec}_{\mathbb{S}}(L) : h \longmapsto h \\ \psi_2 : \text{Spec}_{\wedge}(L) &\longrightarrow \Sigma L : p \longmapsto \left(\phi_p : L \rightarrow \mathbb{S} : l \mapsto \begin{cases} 0 & \text{if } l \leq p \\ 1 & \text{otherwise} \end{cases} \right) \\ \psi_3 : \Sigma L &\longrightarrow \text{Spec}_{\wedge}(L) : h \longmapsto \sup h^{-1}(0) \end{aligned}$$

The fact that ψ_2 is a homeomorphism with inverse ψ_3 can be found in [4,8].

ΣL and $\text{Spec}_{\mathbb{S}}(L)$ are homeomorphic since for any $l \in L$, ev_l is continuous if and only if $\text{ev}_l^{-1}(1) = \Sigma_l$ is open. Hence $\{\Sigma_a \mid a \in L\}$ is coarser than the topology on $\text{Spec}_{\mathbb{S}}(L)$. Since the topology on $\text{Spec}_{\mathbb{S}}(L)$ is the coarsest making the evaluation maps continuous, both topologies are equal. \square

2.4. Spatial \mathbb{F} -frames

Let L be an \mathbb{F} -frame. There is a natural \mathbb{F} -frame homomorphism $\rho_L : L \longrightarrow \mathcal{O}_{\mathbb{F}}\text{Spec}_{\mathbb{F}}(L)$ given by

$$\rho_L : L \longrightarrow \text{Top}(\text{Frm}_{\mathbb{F}}(L, \mathbb{F}), \mathbb{F}) : l \longmapsto (f \mapsto f(l)). \quad (2.1)$$

Definition 2.14. An \mathbb{F} -frame L is called \mathbb{F} -*spatial* if ρ_L is an isomorphism of \mathbb{F} -frames.

Example 2.15. Let L be a frame considered as an \mathbb{S} -frame. Then L is a spatial \mathbb{S} -frame if and only if it is a spatial frame in the usual sense, that is if and only if the following condition holds: for every $a \not\leq b$ in L , there exists $\xi \in \text{Spec}_{\wedge}(L)$ with $a \not\leq \xi$ and $b \leq \xi$.

2.5. \mathbb{F} -sober spaces

Let X and Y be topological spaces.

Definition 2.16.

- (1) X is Y -*initial* if the source of all continuous functions $(f : X \longrightarrow Y)_f$ is initial.
- (2) X is Y -*separated* if the source of all continuous functions $(f : X \longrightarrow Y)_f$ separates points.

Lemma 2.17. Let X , Y and Z be topological spaces.

- (1) If X is Y -initial and Y is Z -initial, then X is Z -initial.
- (2) If X is Y -separated and Y is Z -separated, then X is Z -separated.

Let \mathbb{F} be a topological frame and let X be a topological space.

There is a natural continuous map $\eta_X : X \longrightarrow \text{Spec}_{\mathbb{F}}\mathcal{O}_{\mathbb{F}}(X)$ given by

$$\eta_X : X \longmapsto \text{Frm}_{\mathbb{F}}(\text{Top}(X, \mathbb{F}), \mathbb{F}) : x \longmapsto (f \mapsto f(x)). \quad (2.2)$$

Definition 2.18.

- (1) X is \mathbb{F} -*sober* if η_X is a homeomorphism.
- (2) X is \mathbb{F} -*generating* if every frame homomorphism $\psi : \text{Top}(X, \mathbb{F}) \rightarrow \mathbb{F}$ which satisfies $\psi(c_a) = a$ for every constant function $c_a : X \rightarrow \mathbb{F} : x \mapsto a$ is given by the evaluation function $\psi = \text{ev}_x : \text{Top}(X, \mathbb{F}) \rightarrow \mathbb{F} : f \mapsto f(x)$ of a point $x \in X$.

Proposition 2.19. The following are equivalent:

- (1) X is \mathbb{F} -sober;
- (2) X is \mathbb{F} -initial, \mathbb{F} -separated and \mathbb{F} -generating.

Proof. First assume that X is \mathbb{F} -sober. Hence $\eta_X : x \mapsto \text{ev}_x$ is a homeomorphism. Let $g : Y \rightarrow X$ and $f : X \rightarrow \mathbb{F}$ be continuous. Consider the following diagram.

$$\begin{array}{ccccc}
 Y & \xrightarrow{g} & X & \xrightarrow{f} & \mathbb{F} \\
 & & \downarrow \eta_X & \nearrow \text{ev}_f & \\
 & & \text{Frm}_{\mathbb{F}}(\text{Top}(X, \mathbb{F}), \mathbb{F}) & &
 \end{array}$$

For $x \in X$, we have that $(\text{ev}_f \circ \eta_X)(x) = \text{ev}_f(\eta_X(x)) = \text{ev}_x(f) = f(x)$ and hence the diagram commutes. For any f , since $f \circ g$ is continuous, $\text{ev}_f \circ \eta_X \circ g$ is continuous. By initiality of the source of all $(\text{ev}_f)_f$, $\eta_X \circ g$ is continuous as well and since η_X is a homeomorphism, g is continuous. Hence X is \mathbb{F} -initial.

Let $x \neq y$ be in X . Then $\text{ev}_x = \eta_X(x) \neq \eta_X(y) = \text{ev}_y$. Hence an $f \in \text{Top}(X, \mathbb{F})$ exists such that $f(x) = \text{ev}_x(f) \neq \text{ev}_y(f) = f(y)$. So X is \mathbb{F} -separated.

Finally let $\psi \in \text{Frm}_{\mathbb{F}}(\text{Top}(X, \mathbb{F}), \mathbb{F})$. Then $\psi = \eta_X(x)$ for some $x \in X$, and therefore $\psi = \text{ev}_x$ for some $x \in X$.

Now assume that X is \mathbb{F} -initial, \mathbb{F} -separated and \mathbb{F} -generating. Then η_X is injective since X is \mathbb{F} -separated. Indeed, for $x \neq y$ in X , we can find a continuous $f : X \rightarrow \mathbb{F}$ with $f(x) \neq f(y)$. So $\text{ev}_f(\eta_X(x)) \neq \text{ev}_f(\eta_X(y))$ and hence $\eta_X(x) \neq \eta_X(y)$. For surjectivity, let $\psi \in \text{Frm}_{\mathbb{F}}(\text{Top}(X, \mathbb{F}), \mathbb{F})$. Since X is \mathbb{F} -generating, some $x \in X$ exists with $\psi = \text{ev}_x = \eta_X(x)$.

Continuity of η_X follows since, for any $f \in \text{Top}(X, \mathbb{F})$, $\text{ev}_f \circ \eta_X = f$ and the source of all $(\text{ev}_f)_f$ is initial. For continuity of η_X^{-1} , we use the fact that $\text{ev}_f = f \circ \eta_X^{-1}$ and that X is \mathbb{F} -initial. \square

Let $\mathbb{S} \rightarrow \mathbb{F}$ be the unique frame homomorphism. Recall that an embedding of topological spaces is by definition an initial injective map.

Lemma 2.20. *If $\mathbb{S} \rightarrow \mathbb{F}$ is an embedding, then every topological space is \mathbb{F} -initial and every T_0 -space is \mathbb{F} -separated.*

Proof. Immediate from Lemma 2.17 since all topological spaces are \mathbb{S} -initial and the T_0 spaces are precisely the \mathbb{S} -separated spaces. \square

It is not hard to show that the functor $\mathcal{O}_{\mathbb{F}} : \text{Top} \rightarrow \mathbb{F}/\text{Frm}^{\text{op}}$ is right adjoint to $\text{Spec}_{\mathbb{F}} : \mathbb{F}/\text{Frm}^{\text{op}} \rightarrow \text{Top}$, with the unit of the adjunction determined by (2.2) and the co-unit by (2.1) (see also section 3.3).

3. An investigation of \mathbb{F} -soberness

Let \mathbb{F} be a topological frame and let X be a topological space. The aim of this section is to identify some natural circumstances in which X is \mathbb{F} -sober and to provide some examples of \mathbb{F} -spectra of \mathbb{F} -frames.

3.1. \mathbb{F} -soberness and Hausdorff spaces

Lemma 3.1. *If X is a T_0 space, $0 \neq 1$ in \mathbb{F} and $\{0\} \subseteq \mathbb{F}$ is a closed set, then X is \mathbb{F} -initial and \mathbb{F} -separated.*

Proof. Immediate from Lemma 2.20 since the canonical frame homomorphism $\mathbb{S} \rightarrow \mathbb{F}$ is an embedding in this case. \square

Next, we will investigate circumstances in which X is \mathbb{F} -generating. Consider the frame $L = \text{Top}(X, \mathbb{F})$. We will make use of the canonical map

$$\text{Spec}_{\mathbb{F}}(L) \times \text{Spec}_{\mathbb{S}}(\mathbb{F}) \rightarrow \text{Spec}_{\mathbb{S}}(L)$$

given by composition of frame homomorphisms.

We start by calculating $\text{Spec}_{\mathbb{S}}(L)$. An element $f \in L$ is \wedge -irreducible if $f \neq c_1$ and if $f = g \wedge h$ implies $f = g$ or $f = h$. Let $\text{Spec}_{\wedge}(L)$ denote the set of \wedge -irreducible elements. To a \wedge -irreducible element $p \in L$, we associate the frame homomorphism

$$\phi_p : L \longrightarrow [2] : l \longmapsto \begin{cases} 0 & \text{if } l \leq p \\ 1 & \text{otherwise} \end{cases}$$

and the resulting $\phi : \text{Spec}_{\wedge}(L) \rightarrow \text{Spec}_{\mathbb{S}}(L)$ is a homeomorphism by Proposition 2.13.

For $x \in X$ and $\alpha \in \mathbb{F}$, consider the function

$$f_{x,\alpha} : X \longrightarrow \mathbb{F} : y \longmapsto \begin{cases} \alpha & \text{if } y = x \\ \top & \text{otherwise.} \end{cases}$$

Lemma 3.2. *Suppose X is a Hausdorff space and every closed set $C \subseteq \mathbb{F}$ with $\top \in C$ satisfies $C = \mathbb{F}$. Then we have*

$$\text{Spec}_{\wedge}(L) = \{f_{x,\alpha} \mid x \in X, \alpha \in \text{Spec}_{\wedge}(\mathbb{F})\}. \quad (3.1)$$

Proof. The function $f_{x,\alpha}$ is readily seen to be continuous and \wedge -irreducible. Consider $c_{\top} \neq f \in L$ and suppose $f \neq f_{x,\alpha}$ for all x and α . Hence, there are two points $x \neq y$ in X with $f(x) = \alpha \neq \top$ and $f(y) = \beta \neq \top$. Take disjoint open subsets U_x and U_y with $x \in U_x$ and $y \in U_y$. For $z \in \{x, y\}$, define the function

$$f_z : X \longrightarrow [0, \infty] : a \longmapsto \begin{cases} f(a) & \text{if } a \notin U_z \\ 1 & \text{otherwise} \end{cases}$$

For a closed set $C \subseteq \mathbb{F}$ with $\top \notin C$, we have $f_z^{-1}(C) = f^{-1}(C) \cap (X \setminus U_z)$, hence the function f_z is continuous. Clearly, we have $f = f_x \wedge f_y$ and $f \neq f_x, f \neq f_y$. \square

From now on, we suppose (3.1) holds. For $p = f_{x,\alpha}$, the corresponding element of $\text{Spec}_{\mathbb{S}}(L)$ is

$$\rho_{x,\alpha} = \phi_p : L \longrightarrow [2] : f \longmapsto \begin{cases} 0 & \text{if } f(x) \leq \alpha \\ 1 & \text{otherwise.} \end{cases}$$

Further, for $\beta \in \text{Spec}_{\wedge}(\mathbb{F})$, the corresponding element in $\text{Spec}_{\mathbb{S}}(\mathbb{F})$ is

$$\varphi_{\beta} : \mathbb{F} \longrightarrow [2] : \xi \longmapsto \begin{cases} 0 & \text{if } \xi \leq \beta \\ 1 & \text{otherwise.} \end{cases}$$

Now suppose $\psi : L \longrightarrow \mathbb{F}$ is a frame homomorphism with $\psi(c_a) = a$. For every $\beta \in \text{Spec}_{\wedge}(\mathbb{F})$, we obtain a composed frame homomorphism $\varphi_{\beta}\psi : L \longrightarrow [2]$ and hence an $x \in X$ and $\alpha \in \text{Spec}_{\wedge}(\mathbb{F})$ with

$$\varphi_{\beta}\psi = \rho_{x,\alpha}.$$

Let us prove first that $\alpha = \beta$. For $\gamma \in \text{Spec}_{\wedge}(\mathbb{F})$, we have $\varphi_{\beta}\psi(c_{\gamma}) = \varphi_{\beta}(\gamma) = 0$ if and only if $\gamma \leq \beta$ and $\rho_{x,\alpha}(c_{\gamma}) = 0$ if and only if $\gamma \leq \alpha$. It follows that indeed $\alpha = \beta$. Hence, for every $\beta \in \text{Spec}_{\wedge}(\mathbb{F})$, we obtain $x_{\beta} \in X$ with

$$\varphi_{\beta}\psi = \rho_{x_{\beta},\beta}. \quad (3.2)$$

Lemma 3.3. Consider $\alpha, \beta \in \text{Spec}_\wedge(\mathbb{F})$. If $\alpha < \beta$, then $x_\alpha = x_\beta$.

Proof. We have $\psi^{-1}([0, \alpha]) \subseteq \psi^{-1}([0, \beta])$. Now $f_{x_\alpha, \alpha}$ satisfies $f_{x_\alpha, \alpha}(x_\alpha) \leq \alpha$, hence $\rho_{x_\alpha, \alpha}(f_{x_\alpha, \alpha}) = 0$. So $\varphi_\alpha \psi(f_{x_\alpha, \alpha}) = 0$, which means that $\psi(f_{x_\alpha, \alpha}) \leq \alpha$. So $\psi(f_{x_\alpha, \alpha}) \leq \beta$, $\varphi_\beta \psi(f_{x_\alpha, \alpha}) = 0$ and hence $\rho_{x_\beta, \beta}(f_{x_\alpha, \alpha}) = 0$. Thus $f_{x_\alpha, \alpha}(x_\beta) \leq \beta$. Since $f_{x_\alpha, \alpha}(y) = 1$ for $y \neq x_\alpha$, we necessarily have $x_\beta = x_\alpha$ as desired. \square

Lemma 3.4. Suppose \mathbb{F} is an \mathbb{S} -spatial frame. If two frame homomorphisms $\psi, \phi : L \rightarrow \mathbb{F}$ satisfy

$$\psi^{-1}([0, \gamma]) = \phi^{-1}([0, \gamma])$$

for every $\gamma \in \text{Spec}_\wedge(\mathbb{F})$, then we have $\psi = \phi$.

Proof. Suppose there exists $x \in X$ with $\psi(x) \neq \phi(x)$. Then we have for instance $\psi(x) \not\leq \phi(x)$. Since \mathbb{F} is spatial, there exists a \wedge -irreducible element γ with $\phi(x) \leq \gamma$ and $\psi(x) \not\leq \gamma$, that is, $x \in \phi^{-1}([0, \gamma])$ and $x \notin \psi^{-1}([0, \gamma])$. This contradicts the hypothesis. \square

By (3.2), for $\beta \in \text{Spec}_\wedge(\mathbb{F})$ we have

$$\psi^{-1}([0, \beta]) = \{f \in L \mid f(x_\beta) \leq \beta\} = \text{ev}_{x_\beta}^{-1}([0, \beta]). \quad (3.3)$$

A poset P is called *connected* if for every $p, q \in P$, there exists a finite chain of comparisons

$$p = p_0 < p_1 > p_2 < \cdots < p_{n-1} > p_n = q$$

possibly starting with $p_0 > p_1$ and or $p_{n-1} < p_n$.

Theorem 3.5. Let X be a Hausdorff topological space, and let \mathbb{F} be topological frame such that the following conditions are fulfilled:

- (1) $0 \neq 1$ in \mathbb{F} ;
- (2) \mathbb{F} is spatial;
- (3) $\text{Spec}_\wedge(\mathbb{F})$ is a connected poset;
- (4) $\{0\} \subseteq \mathbb{F}$ is closed;
- (5) $1 \in C \subseteq \mathbb{F}$ and C closed implies $C = \mathbb{F}$.

Then X is \mathbb{F} -sober.

Proof. By Lemma 3.1, X is \mathbb{F} -initial and \mathbb{F} -separated. We show that X is also \mathbb{F} -generating. By Lemma 3.2, (3.1) holds, and hence, for every $\beta \in \text{Spec}_\wedge(\mathbb{F})$, we obtain $x_\beta \in X$ with (3.2). Pick any fixed $\alpha \in \text{Spec}_\wedge(\mathbb{F})$ and put $x = x_\alpha$. Since $\text{Spec}_\wedge(\mathbb{F})$ is connected, for any $\beta \in \text{Spec}_\wedge(\mathbb{F})$, we have $x_\beta = x$ by Lemma 3.3. Thus, by (3.3), we have

$$\psi^{-1}([0, \beta]) = \text{ev}_x^{-1}([0, \beta])$$

for every $\beta \in \text{Spec}_\wedge(\mathbb{F})$. By Lemma 3.4, it follows that $\psi = \text{ev}_x$, as desired. \square

Remark 3.6.

- (1) Condition (3) is fulfilled if $0 \in \text{Spec}_\wedge(\mathbb{F})$.

(2) Conditions (4) and (5) are fulfilled if \mathbb{F} is endowed with the Scott topology.

Corollary 3.7. *Let X be a Hausdorff topological space. Let \mathbb{F} be a chain with $0 \neq 1$ endowed with the Scott topology. Then X is \mathbb{F} -sober.*

Example 3.8. Let $\mathbb{F} = [2] \times [2]$ endowed with the Scott topology. This is seen to be a topological frame. Note that \mathbb{F} satisfies all conditions in Theorem 3.5 except (3): indeed, we have $\text{Spec}_\wedge(\mathbb{F}) = \{(0, 1), (1, 0)\}$ with the pointwise order. Let X be a two point discrete topological space. We have $L = \text{Top}(X, \mathbb{F}) \cong \mathbb{F} \times \mathbb{F}$. Elements in $\text{Spec}_\mathbb{F}(L)$ are frame homomorphisms $\psi : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ with $\psi(a, a) = a$ for all $a \in \mathbb{F}$. The two evaluation functions correspond to the two projections $p_1, p_2 : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$. Now, we see that there exist other elements in $\text{Spec}_\mathbb{F}(L)$, namely the frame homomorphisms

$$\psi_1 : ([2] \times [2]) \times ([2] \times [2]) \longrightarrow [2] \times [2] : ((x, y), (x', y')) \longmapsto (x, y')$$

and

$$\psi_1 : ([2] \times [2]) \times ([2] \times [2]) \longrightarrow [2] \times [2] : ((x, y), (x', y')) \longmapsto (x', y).$$

3.2. Some examples of \mathbb{F} -spectra

In this section, consider, for $n \in \mathbb{N}$, the chain $\mathbf{n} = \{0, 1, \dots, n-1\}$ and $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ endowed with the Scott topology (these are the finite ordinals and $\omega + 1$) and $\mathbb{P} = [0, \infty]$, also with the Scott topology. These are all topological frames.

Proposition 3.9. *For $X = \mathbb{S}$ and $\mathbb{F} = \mathbb{P}$, we have that $\text{Spec}_\mathbb{P}(\text{Top}(\mathbb{S}, \mathbb{P}))$ is homeomorphic to \mathbb{P} .*

Proof. First we note that

$$\begin{aligned} \text{Top}(\mathbb{S}, \mathbb{P}) &= \{f : \mathbf{2} \rightarrow [0, \infty] \mid f(0) \leq f(1)\} \\ &\cong \{(x, y) \in \mathbb{P} \times \mathbb{P} \mid x \leq y\}. \end{aligned}$$

An element $\varphi \in \text{Spec}_\mathbb{P}(\text{Top}(\mathbb{S}, \mathbb{P}), \mathbb{P})$ can now be seen as a map with $\varphi(x, x) = x$ for all $x \in \mathbb{P}$. Since, for $x, y \in \mathbb{P}$ with $x \leq y$,

$$(x, y) = (0, y) \vee (x, x) = (y, y) \wedge (x, \infty),$$

we have that

$$\begin{aligned} \varphi(x, y) &= \varphi(0, y) \vee \varphi(x, x) = \varphi(0, y) \vee x \\ &= (\varphi(y, y) \wedge \varphi(0, \infty)) \vee x = (y \wedge \varphi(0, \infty)) \vee x. \end{aligned}$$

So the map

$$\Phi : \text{Spec}_\mathbb{P}(\text{Top}(\mathbb{S}, \mathbb{P})) \rightarrow \mathbb{P} : \varphi \mapsto \varphi(0, \infty)$$

is injective. We will now show that is surjective. For $\alpha \in \mathbb{P}$, define

$$\varphi_\alpha : \text{Top}(\mathbb{S}, \mathbb{P}) \rightarrow \mathbb{P} : (x, y) \mapsto (y \wedge \alpha) \vee x.$$

Clearly φ_α maps constant functions to their constant value. It also preserves arbitrary joins. Let (x_i, y_i) be in $\mathbf{Top}(\mathbb{S}, \mathbb{P})$ for $i \in I$. Then

$$\begin{aligned} \varphi_\alpha \left(\bigvee_{i \in I} (x_i, y_i) \right) &= \varphi_\alpha \left(\left(\bigvee_{i \in I} x_i, \bigvee_{j \in I} y_j \right) \right) \\ &= \left(\bigvee_{j \in I} y_j \wedge \alpha \right) \vee \bigvee_{i \in I} x_i \\ &= \bigvee_{j \in I} (y_j \wedge \alpha) \vee \bigvee_{i \in I} x_i \\ &= \bigvee_{i, j \in I} (y_j \wedge \alpha) \vee x_i \\ &= \bigvee_{i \in I} (y_i \wedge \alpha) \vee x_i = \bigvee_{i \in I} \varphi_\alpha(x_i, y_i), \end{aligned}$$

where the last equality holds since the join operator is commutative. It also follows that φ_α is increasing.

For finite meets, let $(x, y), (x', y') \in \mathbf{Top}(\mathbb{S}, \mathbb{P})$, we have that $(x \wedge y' \wedge \alpha) \vee (x' \wedge y \wedge \alpha) \leq y \wedge y' \wedge \alpha$ and thus

$$\begin{aligned} \varphi_\alpha(x, y) \wedge \varphi_\alpha(x', y') &= ((y \wedge \alpha) \vee x) \wedge ((y' \wedge \alpha) \vee x') \\ &= (y \wedge y' \wedge \alpha) \vee (y \wedge \alpha \wedge x') \vee (x \wedge y' \wedge \alpha) \vee (x \wedge x') \\ &= (y \wedge y' \wedge \alpha) \vee (x \wedge x') \\ &= \varphi_\alpha((x, y) \wedge (x', y')). \end{aligned}$$

Since \mathbb{S} is \mathbb{P} -initial, Φ and Φ^{-1} are continuous. \square

Proposition 3.10. $\text{Spec}_{\mathbf{n}}(\mathbf{Top}(\mathbb{S}, \mathbf{n}))$ is homeomorphic to \mathbf{n} .

Proof. This is an easy adaptation of the proof of Proposition 3.9. \square

Proposition 3.11. For $X = \mathbf{3}$ and $\mathbb{F} = \mathbb{P}$, we have that $\text{Spec}_{\mathbb{P}}(\mathbf{Top}(\mathbf{3}, \mathbb{P}))$ is homeomorphic to $\{(\alpha, \beta) \in \mathbb{P} \times \mathbb{P} \mid \alpha \geq \beta\}$.

Proof. First we note that

$$\begin{aligned} \mathbf{Top}(\mathbf{3}, \mathbb{P}) &= \{f : \mathbf{3} \rightarrow [0, \infty] \mid f(0) \leq f(1) \leq f(2)\} \\ &\cong \{(x, y, z) \in \mathbb{P}^3 \mid x \leq y \leq z\} \end{aligned}$$

and that

$$\mathbf{Top}(\mathbf{2}^{\text{op}}, \mathbb{P}) = \{f : \mathbf{2} \rightarrow [0, \infty] \mid f(0) \geq f(1)\} \cong \{(\alpha, \beta) \in \mathbb{P} \times \mathbb{P} \mid \alpha \geq \beta\}$$

(where is also endowed with the Scott topology). An element $\varphi \in \text{Spec}_{\mathbb{P}}(\mathbf{Top}(\mathbf{3}, \mathbb{P}), \mathbb{P})$ can now be seen as a map with $\varphi(x, x, x) = x$ for all $x \in \mathbb{P}$. Using the same reasoning as in Proposition 3.9, we can see that for $x, y, z \in \mathbb{P}$ with $x \leq y \leq z$,

$$\varphi(x, y, z) = (z \wedge \varphi(0, 0, \infty)) \vee (y \wedge \varphi(0, \infty, \infty)) \vee x.$$

So the map

$$\Phi : \text{Spec}_{\mathbb{P}}(\text{Top}(\mathbf{3}, \mathbb{P})) \rightarrow \text{Top}(\mathbf{2}^{\text{op}}, \mathbb{P}) : \varphi \mapsto (\varphi(0, \infty, \infty), \varphi(0, 0, \infty))$$

is injective. We will now show that is surjective. For $\alpha, \beta \in \mathbb{P}$ with $\alpha \leq \beta$, define

$$\varphi_{\alpha, \beta} : \text{Top}(\mathbf{3}, \mathbb{P}) \rightarrow \mathbb{P} : (x, y, z) \mapsto x \vee (y \wedge \beta) \vee (z \wedge \alpha).$$

Clearly φ_{α} maps constant functions to their constant value. It also preserves arbitrary joins as in Proposition 3.9.

For finite meets, let $(x, y, z), (x', y', z') \in \text{Top}(\mathbf{3}, \mathbb{P})$, we have that $(x \wedge z' \wedge \alpha) \vee (x' \wedge z \wedge \alpha) \vee (y' \wedge z \wedge \alpha) \vee (y \wedge z' \wedge \alpha) \leq z \wedge z' \wedge \alpha$ and $(x \wedge y' \wedge \beta) \vee (x' \wedge y \wedge \beta) \leq y \wedge y' \wedge \beta$ and thus

$$\begin{aligned} \varphi_{\alpha, \beta}(x, y, z) \wedge \varphi_{\alpha, \beta}(x', y', z') &= (x \vee (y \wedge \beta) \vee (z \wedge \alpha)) \wedge (x' \vee (y' \wedge \beta) \vee (z' \wedge \alpha)) \\ &= (x \wedge x') \vee (y \wedge y' \wedge \beta) \vee (z \wedge z' \wedge \alpha) \\ &= \varphi_{\alpha, \beta}((x, y, z) \wedge (x', y', z')). \end{aligned}$$

Recall that the topology on $\text{Spec}_{\mathbb{P}}(\text{Top}(\mathbf{3}, \mathbb{P}), \mathbb{P})$ is the initial topology for the source

$$(\text{ev}_{(x, y, z)} : \text{Frm}_{\mathbb{P}}(\text{Top}(\mathbf{3}, \mathbb{P}), \mathbb{P}) \rightarrow \mathbb{P} : \varphi \mapsto \varphi(x, y, z))_{(x, y, z) \in \text{Top}(\mathbf{3}, \mathbb{P})} \quad (3.4)$$

and the topology on $\text{Top}(\mathbf{2}^{\text{op}}, \mathbb{P})$ is initial for the source

$$(\text{pr}_i : \text{Top}(\mathbf{2}^{\text{op}}, \mathbb{P}) \rightarrow \mathbb{P})_{i \in \mathbf{2}}. \quad (3.5)$$

Since the maps

$$\text{pr}_0 \circ \Phi = \text{ev}_{(0, 0, \infty)} : \text{Frm}_{\mathbb{P}}(\text{Top}(\mathbf{3}, \mathbb{P}), \mathbb{P}) \rightarrow \mathbb{P} : \varphi \mapsto \varphi(0, 0, \infty)$$

and $\text{pr}_1 \circ \Phi = \text{ev}_{(0, \infty, \infty)}$ are in (3.4), both maps are continuous. Since (3.5) is initial, Φ is continuous.

Now, for $(x, y, z) \in \text{Top}(\mathbf{3}, \mathbb{P})$ and $\gamma \in \mathbb{P}$, consider

$$O_{(x, y, z)}(\gamma) = (\text{ev}_{(x, y, z)} \circ \Phi^{-1})^{-1}([\gamma, \infty]).$$

If we can prove that $O_{(x, y, z)}(\gamma)$ is open in $\text{Top}(\mathbf{2}^{\text{op}}, \mathbb{P})$, then $\text{ev}_{(x, y, z)} \circ \Phi^{-1}$ is continuous and by initiality of (3.4), Φ^{-1} is continuous.

For $(\alpha, \beta) \in \text{Top}(\mathbf{2}^{\text{op}}, \mathbb{P})$, we clearly have that

$$(\alpha, \beta) \in O_{(x, y, z)}(\gamma) \Leftrightarrow x > \gamma \text{ or } y \wedge \beta > \gamma \text{ or } z \wedge \alpha > \gamma.$$

If $\gamma < x$, then $O_{(x, y, z)}(\gamma) = \text{Top}(\mathbf{2}^{\text{op}}, \mathbb{P})$ and if $z \leq \gamma$, then $O_{(x, y, z)}(\gamma) = \emptyset$. So we can assume that $x \leq \gamma < z$. If $x \leq \gamma < y \leq z$, then $(\alpha, \beta) \in O_{(x, y, z)}(\gamma)$ if and only if $y \wedge \beta > \gamma$ or $z \wedge \alpha > \gamma$, which is equivalent to saying that $\beta > \gamma$ or $\alpha > \gamma$. Since $\beta \geq \alpha$, we can reduce this to $\beta > \gamma$. So

$$O_{(x, y, z)}(\gamma) = \text{pr}_0^{-1}([\gamma, \infty]) \cup \text{pr}_1^{-1}([\gamma, \infty]) = \text{pr}_1^{-1}([\gamma, \infty])$$

which is open in $\text{Top}(\mathbf{2}^{\text{op}}, \mathbb{P})$.

If $x \leq y \leq \gamma < z$, then $(\alpha, \beta) \in O_{(x, y, z)}(\gamma)$ if and only if $z \wedge \alpha > \gamma$, which is equivalent to saying that $\alpha > \gamma$. So

$$O_{(x,y,z)}(\gamma) = \text{pr}_0^{-1}([\gamma, \infty])$$

which is open in $\text{Top}(\mathbf{2}^{\text{op}}, \mathbb{P})$. \square

Proposition 3.12. For $X = \mathbb{S} \times \mathbb{S}$ and $\mathbb{F} = \mathbf{3}$, we have that $\text{Spec}_{\mathbf{3}}(\text{Top}(\mathbb{S} \times \mathbb{S}, \mathbf{3}))$ is homeomorphic to $\mathbf{3} \times \mathbf{3}$.

Proof. We will denote $\mathbb{S} \times \mathbb{S} = \{0, a, b, 1\}$ with $0 < a < 1$ and $0 < b < 1$. Then again, we note that

$$\begin{aligned} \text{Top}(\mathbb{S} \times \mathbb{S}, \mathbf{3}) &= \{f : \{0, a, b, 1\} \rightarrow \mathbf{3} \mid f(0) \leq f(a) \leq f(1), f(0) \leq f(b) \leq f(1)\} \\ &\cong \{(\perp, A, B, \top) \in \mathbf{3}^4 \mid \perp \leq A \leq \top, \perp \leq B \leq \top\}, \end{aligned}$$

where we define $\perp = f(0)$, $A = f(a)$, $B = f(b)$ and $\top = f(1)$ in order to simplify notations. An element $\varphi \in \text{Spec}_{\mathbf{3}}(\text{Top}(\mathbb{S} \times \mathbb{S}, \mathbf{3}))$ can now be seen as a map with $\varphi(x, x, x, x) = x$ for all $x \in \mathbb{S} \times \mathbb{S}$. For $(\perp, A, B, \top) \in \text{Top}(\mathbb{S} \times \mathbb{S}, \mathbf{3})$, we have that

$$\begin{aligned} &\varphi(\perp, A, B, \top) \\ &= \varphi((0, 0, 0, \top) \vee (0, A, 0, A) \vee (0, 0, B, B) \vee (\perp, \perp, \perp, \perp)) \\ &= \varphi(((0, 2, 0, 2) \wedge (0, 0, 2, 2) \wedge (\top, \top, \top, \top)) \vee ((0, 2, 0, 2) \wedge (A, A, A, A)) \\ &\quad \vee ((0, 0, 2, 2) \wedge (B, B, B, B)) \vee (\perp, \perp, \perp, \perp)) \\ &= (\varphi(0, 2, 0, 2) \wedge \varphi(0, 0, 2, 2) \wedge \top) \vee (\varphi(0, 2, 0, 2) \wedge A) \vee (\varphi(0, 0, 2, 2) \wedge B) \vee \perp \end{aligned}$$

So the map

$$\Phi : \text{Spec}_{\mathbf{3}}(\text{Top}(\mathbb{S} \times \mathbb{S}, \mathbf{3})) \rightarrow \mathbf{3}^2 : \varphi \mapsto (\varphi(0, 2, 0, 2), \varphi(0, 0, 2, 2))$$

is injective. Using the same techniques as in Proposition 3.11, we can show that it is a homeomorphism. \square

Proposition 3.13. For $X = \mathbf{3}$ and $\mathbb{F} = \mathbf{n}$, we have that $\text{Spec}_{\mathbf{n}}(\text{Top}(\mathbf{3}, \mathbf{n}))$ is homeomorphic to $\{(\alpha, \beta) \in \mathbf{n} \times \mathbf{n} \mid \alpha \geq \beta\}$.

Proof. This is an easy adaptation of the proof of Proposition 3.11. \square

Note that for $L = \text{Top}(\mathbb{S}, \mathbf{3})$ and $\mathbb{F} = \mathbf{3}$, Propositions 3.10 and 3.13 show that for an \mathbb{F} -frame L , $\text{Spec}_{\mathbb{F}}(L)$ is not necessarily \mathbb{F} -sober.

3.3. Relation to sobriety of fuzzy frames

Using fuzzy topology and fuzzy frames, it is also possible to define a more general notion of sobriety, as is done in [16]. We will recall some of the properties needed to compare this to our notion.

Definition 3.14. Let \mathbb{F} be a topological frame. An \mathbb{F} -fuzzy topological space is a pair (X, τ_X) with X a set and τ_X a subframe of $\text{Fun}(X, \mathbb{F})$, containing all constant maps.

A map $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ is continuous if $gf \in \tau_X$ for all $g \in \tau_Y$. The category of \mathbb{F} -topological spaces and continuous maps is denoted by $\mathbb{F}\text{-Top}$.

Now, consider the following functors:

$$\text{Top} \begin{array}{c} \xrightarrow{\omega_{\mathbb{F}}} \\ \xleftarrow{\iota_{\mathbb{F}}} \end{array} \mathbb{F}\text{-Top} \begin{array}{c} \xrightarrow{\Omega_{\mathbb{F}}} \\ \xleftarrow{\text{pt}_{\mathbb{F}}} \end{array} (\mathbb{F}/\text{Frm})^{\text{op}}$$

Here, $\omega_{\mathbb{F}}$ maps (X, \mathcal{T}) to $(X, \mathbf{Top}(X, \mathbb{F}))$ and ι_F maps (X, τ_X) to $(X, \mathcal{T}_{\text{in}})$ where \mathcal{T}_{in} is the initial topology for τ_X . These functors constitute an adjunction (see [5,6,13]). Moreover, if X is \mathbb{F} -initial, $\iota_{\mathbb{F}}\omega_{\mathbb{F}} = 1_{\mathbf{Top}}$.

The functor $\Omega_{\mathbb{F}}$ maps an \mathbb{F} -fuzzy space (X, τ_X) to τ_X and $\text{pt}_{\mathbb{F}}$ maps an \mathbb{F} -frame L to $(\text{Frm}_{\mathbb{F}}(L, \mathbb{F}), \{\text{ev}_l\}_{l \in L})$. As shown in [16], these also constitute an adjunction.

Clearly $\mathcal{O}_{\mathbb{F}} = \Omega_{\mathbb{F}}\omega_{\mathbb{F}}$ and $\text{Spec}_{\mathbb{F}} = \iota_{\mathbb{F}}\text{pt}_{\mathbb{F}}$, proving in particular that these functors are also adjoint as claimed earlier on at the end of section 2.

Definition 3.15. An \mathbb{F} -fuzzy space X is called \mathbb{F} -fuzzy sober if $\text{pt}_{\mathbb{F}}\Omega_{\mathbb{F}}X$ is homeomorphic to X . An \mathbb{F} -frame L is called \mathbb{F} -fuzzy spatial if $\Omega_{\mathbb{F}}\text{pt}_{\mathbb{F}}L$ is isomorphic to L .

The following is Theorem 1.4 in [16].

Theorem 3.16. For an \mathbb{F} -fuzzy space X , $\Omega_{\mathbb{F}}X$ is \mathbb{F} -fuzzy spatial and $\text{pt}_{\mathbb{F}}\Omega_{\mathbb{F}}X$ is \mathbb{F} -fuzzy sober.

We have the following relation:

Proposition 3.17. Suppose X is an \mathbb{F} -initial topological space such that $\omega_{\mathbb{F}}X$ is \mathbb{F} -fuzzy sober. Then X is \mathbb{F} -sober.

Proof. If $f : \omega_{\mathbb{F}}X \rightarrow \text{pt}_{\mathbb{F}}\Omega_{\mathbb{F}}\omega_{\mathbb{F}}X$ is a homeomorphism in $\mathbb{F}\text{-Top}$, $\iota f : X = \iota_{\mathbb{F}}\omega_{\mathbb{F}}X \rightarrow \text{Spec}_{\mathbb{F}}\mathcal{O}_{\mathbb{F}}X$ is a homeomorphism in \mathbf{Top} . \square

Contrary to the situation described in Theorem 3.16, we have that $\text{Spec}_{\mathbb{F}}\mathcal{O}_{\mathbb{F}}X$ is not \mathbb{F} -sober in general. The sobrification fails when going back from $\mathbb{F}\text{-Top}$ to \mathbf{Top} , even if X is \mathbb{F} -initial. To see this, take $\mathbb{F} = \mathbf{3}$. Then $\text{pt}_{\mathbb{F}}\Omega_{\mathbb{F}}\omega_{\mathbb{F}}\mathbf{S}$ is \mathbb{F} -fuzzy sober, but $\iota_{\mathbb{F}}\text{pt}_{\mathbb{F}}\Omega_{\mathbb{F}}\omega_{\mathbb{F}}\mathbf{S}$ is not \mathbb{F} -sober.

Proposition 3.18. Let X be an \mathbb{F} -initial space such that $\text{Spec}_{\mathbb{F}}\mathcal{O}_{\mathbb{F}}X$ is not \mathbb{F} -sober. Then $\text{pt}\Omega_{\mathbb{F}}\omega_{\mathbb{F}}X$ is not homeomorphic to $\omega_{\mathbb{F}}Y$ for any \mathbb{F} -initial space Y .

Proof. Since $\text{Spec}_{\mathbb{F}}\mathcal{O}_{\mathbb{F}}X$ is not \mathbb{F} -sober, X is also not \mathbb{F} -sober and hence $\omega_{\mathbb{F}}X$ is not \mathbb{F} -fuzzy sober. But $W := \text{pt}_{\mathbb{F}}\Omega_{\mathbb{F}}\omega_{\mathbb{F}}X$ is \mathbb{F} -fuzzy sober. Assume that $W = \omega_{\mathbb{F}}Y$ for some Y , then Y is \mathbb{F} -sober. But $Y = \iota_{\mathbb{F}}\omega_{\mathbb{F}}Y = \iota_{\mathbb{F}}W = \text{Spec}_{\mathbb{F}}\mathcal{O}_{\mathbb{F}}X$, which is a contradiction with the assumption that $\text{Spec}_{\mathbb{F}}\mathcal{O}_{\mathbb{F}}X$ is not \mathbb{F} -sober. \square

There are still other notions of sobriety of \mathbb{F} -topological spaces. Some of these are compared in [15]. A specific one worth mentioning is the one by Pultr and Rodabaugh in [11] and later generalised in [2] as they make use of the functors $\omega_{\mathbb{F}}$ and $\iota_{\mathbb{F}}$, contrary to Zhang and Liu. Their version of sobriety for an \mathbb{F} -fuzzy space X is equivalent to the classical sobriety of $\iota_{\mathbb{F}}X$. However, their sobriety is weaker than the one by Zhang and Liu, whereas ours is clearly not, hence both notions are not equivalent.

4. Research questions

To finish this note, we give an outline of some further questions which we are currently investigating and which will be treated in future publications.

4.1. Relation between soberness with respect to different topological frames

In principle, different choices of \mathbb{F} may give rise to new notions of \mathbb{F} -sobriety for a given topological space. For a certain class of topological frames \mathbb{F} , these notions of soberness might be related (at first we

are mainly interested in the class of topological frames containing \mathbb{S}). Can we find a set of conditions to impose on topological frames $\mathbb{F}_1, \mathbb{F}_2$ such that the implications

$$X \text{ Hausdorff} \Rightarrow X \text{ } \mathbb{F}_1\text{-sober} \Rightarrow X \text{ } \mathbb{F}_2\text{-sober} \Rightarrow X \text{ sober}$$

hold?

For $\mathbb{F}_1, \mathbb{F}_2$ in a class of topological frames with additional properties, can we find a general description for $\text{Spec}_{\mathbb{F}_2}(\text{Top}(X, \mathbb{F}_1))$? Or even just for $\mathbb{F}_1 = \mathbb{F}_2$?

4.2. The forgetful functor to frames

For a topological frame \mathbb{F} , there is a natural forgetful functor $U : \mathbb{F}/\mathbf{Frm} \rightarrow \mathbf{Frm} : (L, \gamma_L : \mathbb{F} \rightarrow L) \mapsto L$. What can be said about the image of this functor, and the fiber of a given $L \in \mathbf{Frm}$? There are natural actions of both $\mathbf{Frm}(\mathbb{F}, \mathbb{F})$ and $\mathbf{Frm}(L, L)$ on the set $\Phi_{\mathbb{F}}(L) = \{\gamma_L : \mathbb{F} \rightarrow L\}$ of \mathbb{F} -frame structures on a given frame L .

The fact that, in Example 3.8, the frame $\mathbb{F} = \mathbb{S} \times \mathbb{S}$ gives rise to a different notion of \mathbb{F} -soberness seems related to the inner automorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ which exchanges the elements $(0, 1)$ and $(1, 0)$. If this is indeed the case, can we somehow “mod out” the action of this automorphism?

4.3. Relation with EM-categories

The category \mathbf{Frm} can be obtained as an Eilenberg-Moore category from a standard adjunction between \mathbf{Top}^{op} and \mathbf{Set} (see e.g. [4] and also [12] in this respect). Does a similar setup exist for \mathbb{F}/\mathbf{Frm} ?

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